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GAMES WITH CIRCULAR SYMMETRY

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GAMES WITH CIRCULAR SYMMETRY

L. J. Savage

Games defined by a pay-off function

$$P(m,n) = \int_0^{2\pi} \int_0^{2\pi} f(x-y)dm(x)dn(y), \tag{1}$$

where $f(z)$ is bounded, measurable, of period 2π, and where m and n are probability measures on the interval from 0 to 2π, have been of occasional interest to some of us at RAND. The purpose of this memorandum is to characterize the effective strategies of such a game.

The x-player can, with the strategies $dm(x) = dx/2\pi$, win

$$\begin{aligned} P(m,n) &= \iint f(x-y)\frac{dx}{2\pi}\, dn(y) \\ &= \int dn(y) \frac{1}{2\pi} \int dz f(z) \\ &= \frac{1}{2\pi} \int f(z)dz = \phi_0 . \end{aligned} \tag{2}$$

Similarly the y-player can with the strategy $dn(y) = dy/2\pi$ lose exactly ϕ_0, independently of m. Therefore, the value of the game is ϕ_0, and the strategies $dx/2\pi$ and $dy/2\pi$ are effective for the two players.

Are there other effective strategies, and if so, do they also yield an income altogether independent of the behavior of the opponent? Fourier analysis leads naturally to the answer. The function f and the measures m and n are characterized by their Fourier series coefficients ϕ_k, μ_k, and ν_k respectively.

(3) $$\phi_k = \frac{1}{2\pi}\int f(z)e^{ikz}dz;$$

(4) $$\mu_k = \int e^{ikz}dm(z); \qquad k=0, \pm 1,\ldots$$

(5) $$\nu_k = \int e^{ikz}dn(z).$$

By standard arguments

(6) $$\mu_0 = \nu_0 = 1.$$

(7) $$\phi_k = \bar{\phi}_{-k}; \quad \mu_k = \bar{\mu}_{-k}; \quad \nu_k = \bar{\nu}_{-k}.$$

(8) $$P(m,n) = \sum_{-\infty}^{+\infty} \phi_k \bar{\mu}_k \nu_k .$$

<u>THEOREM 1</u>. The strategy m,(n) is effective if and only if $\phi_k \mu_k = 0$ ($\phi_k \nu_k = 0$) for all $k \neq 0$. If m,(n) is effective, $P(m,n) = \phi_0$ irrespective of n,(m).

<u>Proof:</u> Suppose $\phi_k \mu_k = 0$ ($\phi_k \nu_k = 0$_ for $k \neq 0$, then it follows directly from (6)-(8) that $P(m,n) = \phi_0$.

Consider on the other hand an m such that for some $k' \neq 0$, $\phi_{k'} \mu_{k'} \neq 0$. Then $\phi_{k'} \mu_{k'} = \alpha \neq 0$. Let n be defined by

(9) $$dn(x) = \frac{1}{2\pi}\left\{1 + \frac{1}{2|\alpha|}(\alpha e^{ik'x} + \bar{\alpha}e^{-ik'x})\right\} dx.$$

It is easy to verify that n is a probability measure and that its Fourier coefficients are

(10) $$\nu_0 = 1; \quad \nu_{k'} = \bar{\alpha}/2|\alpha| ; \quad \nu_{-k'} = \alpha/2|\alpha| ;$$

$$\nu_k = 0 \text{ if } k \text{ is neither } 0,\ k' \text{ or } -k'.$$

Therefore $P(m,n) = \phi_0 - |\alpha|$, so m is not effective. Similar consideration of n completes the proof.

THEOREM 2. The strategy $dx/2\pi$, $(dy/2\pi)$ is the only effective strategy for the x-player (y-player) if and only if $\phi_k \neq 0$ when $k \neq 0$.

Proof: The whole theorem follows from Theorem 1 together with the remark that

$$
\begin{aligned}
dm(x) &= \frac{1}{2\pi}\left\{1 + \cos kx\right\} dx \\
dn(y) &= \frac{1}{2\pi}\left\{1 + \cos ky\right\} dy
\end{aligned}
\tag{11}
$$

define probability measures.

Extensions of these results to other highly symmetric pay-off functions suggest themselves.

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